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Could L'Hospital have read Newton's *Methodus Fluxionum*?

1. Introduction

In 1696 the Marquis de L'Hospital published the *Analyse des infiniment petits*,[1] the first systematic educational work on differential calculus. It was based on Johann Bernoulli's *Lectiones de calculo differentialium*, a manuscript containing the lessons Bernoulli gave L'Hospital between 1691 and 1692.[2] However, both works cannot be said to be identical.[3] The most remarkable difference between them concerns the choice of coordinates, and therefore the treatment of algebraic and trancendental curves. The tendency towards algebraization in the eighteenth century entailed an increasing use of orthogonal coordinates, thus announcing the emergence of the concept of function. Johann Bernoulli's *Lectiones* illustrate this tendency, all the more so since the idea of the ordinate y as a function of the abscissa x pervaded his manuscript. In fact the first explicit definition of function as an analytic expression was given by Johann Bernoulli in an article of 1718, "Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes sur les isopérimètres", published in the *Mémoires de l'Académie Royale des Sciences de Paris*.[4]

Quite to the opposite, L'Hospital shew a penchant towards a geometric treatment of the curve, which led him to choose the coordinates according to the geometric nature of the curve. Consequently, the equation of the transcendental curve became clearer and simpler. This was the usual procedure at the end of the seventeenth century.

Taking the choice of coordinates as an indicator of the connection algebra-geometry, the differences between Bernoulli and L'Hospital turned out to be revealing in the analysis of the process of algebraization of calculus, even at the first stage of its development, traditionally regarded as geometric.[5] In this regard I wondered why L'Hospital took coordinates different from Bernoulli's and whether these different viewpoints in selecting the coordinates were communicated. How did L'Hospital appropriate Bernoulli's *Lectiones de calculo differentialium*?

In a broad sense, the goal of my work was to answer these questions from an approach centered on the conceptualisation of science as communication.[6] In this respect, I took into consideration Schubring's views on educational systems as units of communication,[7] especially regarding educational books as emerging from a specific educational system. Besides, the notion of appropriation, that is to say, how the knowledge that circulates is taken up, interpreted and put to use in a specific context, seemed to offer an interesting frame to

[1] G. F. A. de L'Hospital, *Analyse des infiniment petits pour l'intelligence des lignes courbes* (1696).

[2] Johann Bernoulli, *Lectiones de calculo differentialium 1691-1692*. Ed. P. Schafheitlin (1922).

[3] See M. Blanco, "Análisis de la controversia L'Hôpital-Bernoulli", *Cronos* (2001), pp. 81-113.

[4] See A. P. Youschkevitch, "The concept of function up to the middle of the 19th century", *Archive for History of Exact Sciences* (1976), pp. 37-85.

[5] See, for instance, C. G. Fraser, "The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century", *Archive for history of exact sciences* (1998), pp. 317-335.

[6] J. A. Secord, "Knowledge in Transit", *Isis* (2004), pp. 654-72. I started developing this approach in a contribution to the 2nd Meeting for Postgraduate Students in History of Science (Barcelona, 2007): "The Concept of Curve in the Works that Originated from Johann Bernoulli's *Lectiones de Calculo Differentialium*".

[7] G. Schubring, "Changing cultural and epistemological views on mathematics and different institutional contexts in nineteenth-century Europe", In C. Goldstein et al. (eds.): *Mathematical Europe. Myth, History, Identity* (1996), pp. 363-388.

answer the abovementioned questions.[8] Hence, I intended to survey in depth the connection algebra-geometry through some texts on differential calculus, to explore the process of algebraization into the first stage of calculus, to answer how it was communicated and appropriated.

Having set my work in its original framework, the specific aim of the current paper is to analyse the network of communication practices to get some insights on how the choice of coordinates in the problem of the tangent to the cycloid was communicated and appropriated. In particular, this paper opens with a revision of how the problem of the tangent to the cycloid was tackled, providing some valuable information on the way the authors involved regarded the concept of curve. I focus on the tangent to the cycloid since the study of this curve it attracted so many mathematicians in the seventeenth century and on through the eighteenth.

2.Tangent to the Cycloid

2.1 Johann Bernoulli's solution in the *Lectiones de calculo differentialium*

In problem VI of his *Lectiones de calculo differentialium*, Bernoulli's aimed at calculating the subtangent of the cycloid, s. Considering:

$$x=BF,\ y=EF=BM,\ f=EH=\text{arc}(HB),$$

where x and y are orthogonal, he drew EM parallel to AC (Figure 1).

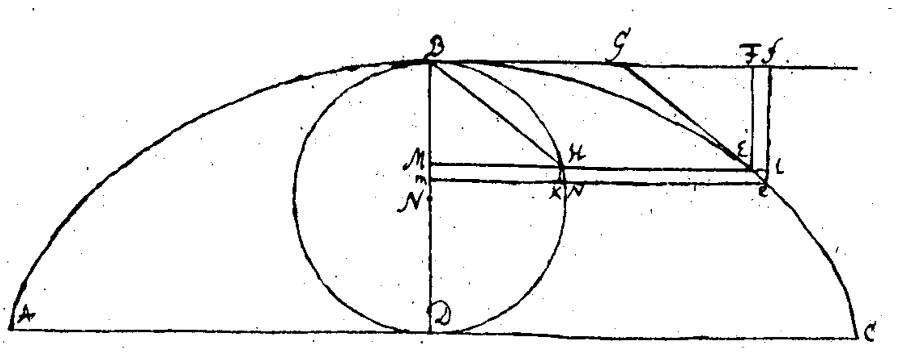

Figure 1. Determination of the subtangent to the cycloid according to Johann Bernoulli

From the cycloid's nature, it follows:

$$x=EH+HM=f+\sqrt{2ay-y^2}\ ,$$

$$dx = df + \frac{2ady - 2ydy}{2\sqrt{2ay - y^2}}.$$

Given that a curve can be considered as a polygon of infinitely many infinitesimal sides,[9] ΔHKN is a rectangular triangle:

[8] On the issue of "appropriation", see: J. R. Topham, "Scientific Publishing and the Reading of Science in Nineteenth-Century Britain: A Historiographical Survey and Guide to Sources", *Studies in History and Philosophy of Science* (2000), pp. 564-566; L. Roberts, "Circulation Promises and Challenges", *Workshop The Circulation of Knowledge and Practices: The Low Countries as an historical laboratory. Woudschoten* (2005).

[9] See the second postulate in J. Bernoulli, *Lectiones*… (1922).

$$HN=\sqrt{HK^2+KN^2}=\frac{ady}{\sqrt{2ay-y^2}},$$

$$dx=\frac{2ady-ydy}{\sqrt{2ay-y^2}}.$$

This approach produces the following differential equation,

$$\frac{dy}{dx}=\frac{y}{s}$$

that yields:

$$s=\frac{2ay-y^2}{\sqrt{2ay-y^2}}=\sqrt{2ay-y^2}=HM$$

2.2. L'Hospital's solution in the *Analyse des infiniment petits*

Section II of the *Analyse des infiniment petits* is devoted to the *Usage du calcul des différences pour trouver les tangentes de toutes sortes de lignes courbes*. L'Hospital opens up *Proposition II* with a general result. So as to determine the tangent *MT* of a given curve, generated by another curve, L'Hospital took the abscissa over the generating curve, $x=AP$, with ordinate $y=PM$, along with an infinitely close ordinate, *pm* (see Figure 2). From *M* he drew the segment *MR* parallel to *PT* (tangent of the generating curve), and such that $MR=Pp=dx$ and $Rm=dy$.

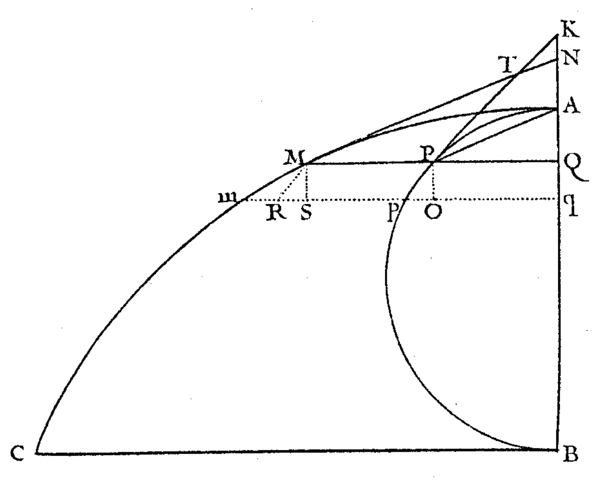

Figure 2. Determination of the tangent to the cycloid according to L'Hospital

From the similarity of the triangles ΔmRM and ΔMPT it follows:

$$\frac{dy}{dx}=\frac{MP}{PT},$$

$$PT=\frac{ydx}{dy}.$$

L'Hospital then proceeded with the determination of the tangent to the cycloid as a particular example. Taking the relationship between the segments *x*, *y* when the curve *APB* is a circle, produces the following equation for the cycloid:

$$x = \frac{ay}{b},$$

L'Hospital identified the elements of the general proposition above with the corresponding segments of the cycloid and this approach produces a differential proportion,

$$dx = \frac{ady}{b},$$

$$PT = \frac{ay}{b} = x.$$

Why did L'Hospital not follow his master in the selection of coordinates for the cycloid? In the correspondence that Johann Bernoulli and the Marquis de L'Hospital exchanged essentially between 1692 and 1695,[10] I came across one and only letter where they discussed the subject of selection of coordinates for the curve generated from a circle rolling over another circle. In the following excerpt of this letter I have underlined the most significant lines regarding this subject:

> C'est je crois la resolution la plus courte et la plus naturelle que je vous ay envoyée pour trouver les plus grandes largeurs des roulettes, vû qu'elle est fondée sur la generation méme de la roulette; or ces sortes de solutions sont toujours preferables aux autres, qui ne sont pas immediatement tirées de la nature du probleme, et qui par consequent n'expliquent pas avec si grande evidence l'essence, en laquelle il consiste. Outre cela les solutions qu'on tire des équations pour la relation des coordonnées des courbes sont ordinairement plus prolixes que les autres qui se trouvent par la generation méme des courbes. Et ainsy je croyois qu'il valloit mieux de vous envoyer la plus courte, quoiqu'en effet je sçache aussy une manier de resoudre le probleme par le moyen de la roulette meme que l'on suppose décrite: je n'ay pas encor fait le calcul parce qu'il me paroit un peu long; je vous indiqueray icy seulement la voye pour y parvenir, la voycy:... il faut chercher l'equation qui exprime la nature de la roulette, c'est à dire la relation entre l'abscisse et l'ordonnée, or cette equation ne peut étre que differentielle parce que la roulette est quelquefois transcendente. (J. Bernoulli, *Briefwechsel*… (1955), letter N. 48, from Johann Bernoulli to L'Hospital)

And further down in the same letter Bernoulli added:

> Mais la premiere maniere que je vous ay envoyée est incomparablement meilleure, parce que la determination de ce point se donne par une simple analogie et sans beaucoup de calcul, au lieu que par cette autre maniere on n'y arriveroit qu'aprez un terrible calcul... ce qui fait assez voir combien il est utile de choisir la voye la plus naturelle en resolvant les problemes. (J. Bernoulli, *Briefwechsel*… (1955), letter N. 48, from Johann Bernoulli to L'Hospital)

That L'Hospital agreed with this view it is clear from his reply: "car il est sans contredit que vôtre construction qui se tire de la generation de la roulette est sans comparaison plus simple que celle qui se tire de la relation des coordonnées" (J. Bernoulli, *Briefwechsel*… (1955), letter N. 49, from L'Hospital to Johann Bernoulli).

The consideration of the generation of the curve related to the transcendental curves reminded me of Newton's solution of the problem. L'Hospital's approach turned out to be similar to the one Isaac Newton presented in his *Methodus Fluxionum*, written in 1671 though

[10] Johann Bernoulli, *Der Briefwechsel von Johann Bernoulli, Herausgegeben von der Naturforschenden Gesellschaft in Basel*. Ed. O. Spiess (1955).

not published until 1736.[11] The statements in L'Hospital's book held a similar structure, that is, first a general proposition, for any two curves; then, some specific examples. The next section summarises the determination of the tangent to the cycloid in Newton's *Methodus Fluxionum*, and points out the possible connection between Newton and L'Hospital in this matter.

2.3. Newton's solution in the *Methodus Fluxionum*

In the ninth manner of the *Methodus Fluxionum* included in Problem IV (*To draw tangents to curves*), Newton introduced how to find the tangent to a curve that is generated by another curve. Hence, it was again a question of finding the tangent. Figure 3 below reproduces Newton's general solution as described in §59.

Ninth Manner.

59. Laſtly, if ABF be any given Curve, which is touch'd by the right Line B*t*; and a part BD of the right Line BC, (being an Ordinate in any given Angle to the Abſciſs AC,) intercepted between this and another Curve DE, has a Relation to the portion of the Curve AB, which is expreſs'd by any Equation: You may draw a Tangent DT to the other Curve, by taking (in the Tangent of this Curve,) BT in the ſame Ratio to BD, as the Fluxion of the Curve AB hath to the Fluxion of the right Line BD.

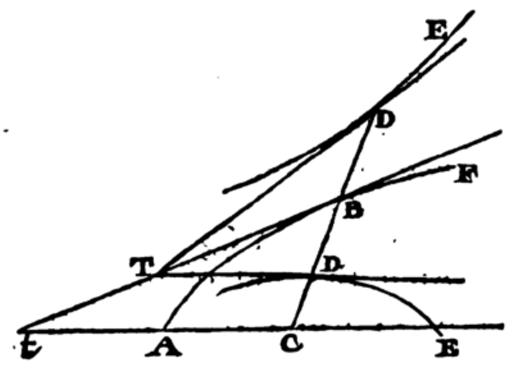

Figure 3. Ninth manner to determine the tangent to a curve, from the English version of Newton's *Methodus Fluxionum*.

Then, in §61 this general statement is applied to the particular case of the "trochoid," when *ABF* is a circle. Hence, both Newton and L'Hospital used the same coordinates, obtaining the (simple) equation $\frac{a}{b}x = y$.

As it is evident from the *Analyse*'s preface, L'Hospital read the *Principia*, the only Newtonian printed work at the time (1687):[12]

> C'est encore une justice dûë au sçavant M. Newton, (...): Qu'il avoit aussi trouvé quelque chose de semblable au Calcul différentiel, comme il paroît para l'excellent Livre intitulé *Philosophia naturalis principia Mathematica*, qu'il nous donna en 1687. lequel est presque tout de ce calcul. (G. F. A. de L'Hospital, *Analyse*… (1696), preface, p. 12)

However, this work contained just one lemma concerning some rules of the Newtonian calculus, not using the characteristic notation, nor mentioning fluxions at all (Lemma II,

[11] I. Newton, *Methodus fluxionum et serierum infinitarum* (1671). This Latin version was published only in 1779. English version: *The Method of fluxions and infinite series* (1736); French version: *La Méthode des fluxions et des suites infinies* (1779).

[12] I. Newton, *Philosophiae naturalis principia mathematica* (1687).

section II, book II). Hence, my first conjecture was whether part of the *Methodus Fluxionum* could have somehow reached L'Hospital between 1691 and 1696 through circulation of manuscripts. From the analysis of the circulation of Newtonian manuscripts I have to admit that it was pretty unlikely.

There was a second more likely possibility: the mutual influence between Newton's and Barrow's work, alongside the fact that L'Hospital read Isaac Barrow's *Lectiones Geometricae* (1670),[13] as it is inferred from the *Analyse*'s preface:

> M. Barrow (*Lect. geomet. p. 80*) n'en demeura pas là, il inventa aussi une espéce de calcul propre à cette méthode; mais il luy falloit, aussi-bien que dans celle de M. Descartes, ôter les fractions & faire évanoüir tous les signes radicaux pour s'en servir.
> Au défaut de ce calcul est survenu celuy du célébre M. Leibnis; & ce Sçavant Géometre à commencé où M. Barrow & les autres avoient fini. (G. F. A. de L'Hospital, *Analyse*… (1696), preface, p. 7)

Additionally, the correspondence involving L'Hospital and Johann Bernoulli, on the one hand, and Christiaan Huygens, on the other, seems to provide some more proofs of Barrow's influence on L'Hospital's work.[14] Thus, in a letter to L'Hospital, Bernoulli claims:

> Je me souviens qu'au comencement du temps que je demeurois à Paris vous me demandiez souvent à quoy bon de se servir dans le calcul differentiel de la lettre characteristique *d*, si on ne pourroit pas mettre à la maniere de Barrow *a* et *e* à la place de *dx* et *dy*... (J. Bernoulli, *Briefwechsel*… (1955), letter N. 59, from Johann Bernoulli to L'Hospital)

The fact that L'Hospital's belonged to Nicolas Malebranche's circle allowed him to correspond with Huygens (1690-1695) as well.[15] From this correspondence it is clear that L'Hospital was aware of Newton's progress, as well as of other British mathematicians' works, such as Isaac Barrow (on the method of tangents) and John Wallis (on Newton's inverse method of tangents).[16] To the purpose of this paper, the next section displays how Barrow determined the tangent to the cycloid in his *Lectiones Geometricae*.

2.4. Barrow's solution in the *Lectiones Geometricae*

In Lecture V (*Further properties of curves. Curves like the cycloid. Normals. Maximum and minimum*), Barrow expands his solution to the problem as it follows:

> 6. A straight line *AY*, moving parallel to itself, traverses any curve, either concave or convex to the same parts, with uniform motion (that is to say, it passes over equal parts of the curves in equal times), and simultaneously any point is carried, also uniformly, along *AY* from *A*; by the point moving in this manner there is generated a curve *AMZ*, of which it is required to find the tangent at any point *M*.
> To do this, draw *MP* parallel to *AY* to cut the curve *APX* in *P*; through *P* draw the straight line *PE* touching the curve *APX*; through *M* draw *MH* parallel to *PE*; take any point *R* in *MH*, and draw *RS* parallel to *PM*; mark off *RS* so that $MR : RS = arcAP : PM$ (*i.e.* as the one uniform motion is to the other); join *MS*. Then *MS* will touch the curve *AMZ*.

[13] J. M. Child, *Geometrical Lectures of Isaac Barrow* (1916).

[14] C. Huygens, *Correspondance de Christiaan Huygens* (1888-).

[15] There is a discussion on the role of Malebranche as the centre of communication network involving the differential calculus in M. Blanco, "On how Johann Bernoulli's lessons on differential calculus were communicated in eighteenth-century France and Italy", *Beyond Borders* (1998). Hence, he could be considered as a "nodal point," as Lux and Cook puts it. See D. S. Lux & H. J. Cook, **"**Closed circles or open networks?: Communicating at a distance during the Scientific revolution", *History of Science* (1998).

[16] See, for instance, letters n. 2775, 2787, 2815, 2843, 2859 and 2879 in C. Huygens, *Correspondance*… (1888-).

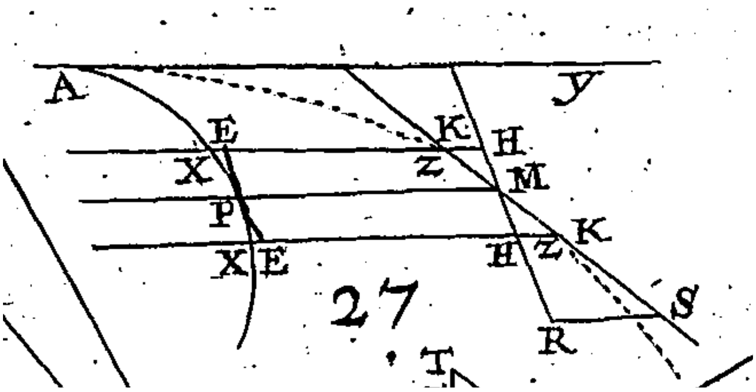

Figure 4. Determination of the tangent to the cycloid according to Barrow

From a previous result in the same lecture, the arc *AP* (assumed to be indefinitely small) can be considered to be equal to the small bit of the tangent *PE*.

Hence, the tangent to the generating curve, *PE*, is here taken into account, as it is later in L'Hospital's solution. In this regard it is also worth reading Lecture X (*Rigorous determination of ds/dx. Differentiation as the inverse of integration. Explanation of the "Differential Triangle" method; with examples. Differentiation of a trigonometrical function*):

> 1. Let *AEG* be any curve whatever, and *AFI* another curve so related to it that, if any straight line *EF* is drawn parallel to a straight line given in position (which cuts *AEG* in *E* and *AFI* in *F*), *EF* is always equal to the arc *AE* of the curve *AEG*, measured from *A*; also let the straight line *ET* touch the curve *AEG* at *E*, and let *ET* be equal to the arc *AE*; join *TF*; then *TF* touches the curve *AFI*.

This result still holds for any the same ratio of *EF* to the arc *AE*.

In short, from the reading of both lectures it stands out the relevant role played not only by the tangent to the generating curve (i.e. a circle), but also by the generating elements of the cycloid, namely, *PM* and the arc *AP*.

3. Some Final Remarks

Regarding the problem of the tangent to the cycloid, it was worth analysing how the Marquis de L'Hospital appropriated Johann Bernoulli's manuscript, all the more so since the choice of coordinates can be considered as an indicator of the process of the algebraization of calculus.

Such an analysis was made possible from the analysis of communication practices involved, in particular, the correspondence of L'Hospital, granted by his connection with Nicolas Malebranche. On the one hand, the correspondence between Johann Bernoulli and L'Hospital sheds some light on their exchanges on the selection of coordinates. On the other hand, the correspondence between Christiaan Huygens and L'Hospital highlights the latter's interest in British mathematics. No doubt was the *Analyse des infiniment petits* largely based on Bernoulli's *Lectiones de calculo differentialium*. Yet, it becomes plain that L'Hospital was also influenced by Isaac Barrow's work and, not unlikely, by Isaac Newton's.

To conclude, in showing L'Hospital's interest in British mathematics, this paper contributes to challenge the "traditional" vision of the lack of exchange between British and continental European mathematics.[17]

[17] For further details on this issue, see G. Schubring, "Changing cultural…" (1996), p. 376.